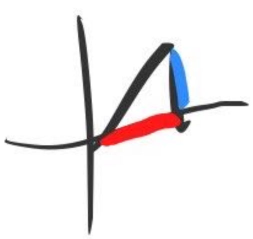
Normed spaces

To do (certain kinds of) geometry you need a way to measure **distance**

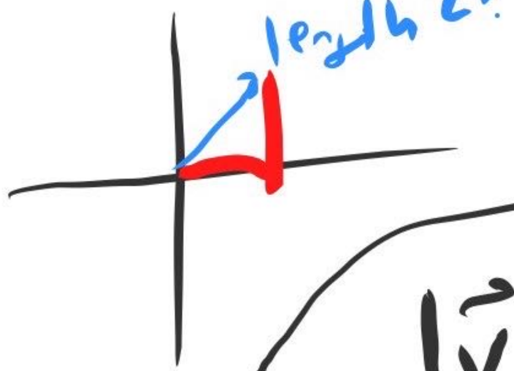
In Calculus (inspired by Pythagoras) we defined the length of $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ to be $|\vec{v}|_2 = \sqrt{v_1^2 + \cdots + v_n^2}$



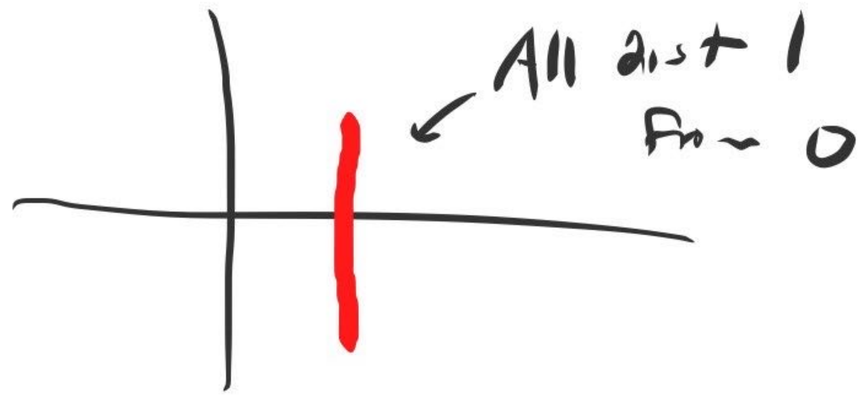
This is hardly the only way to measure distance...

there are, eg, the "taxicab length"

$$|\vec{v}|_1 = |v_1| + \cdots + |v_n|$$



$$|\vec{v}|_\infty = \max(|v_1|, \dots, |v_n|)$$



We need a general notion.

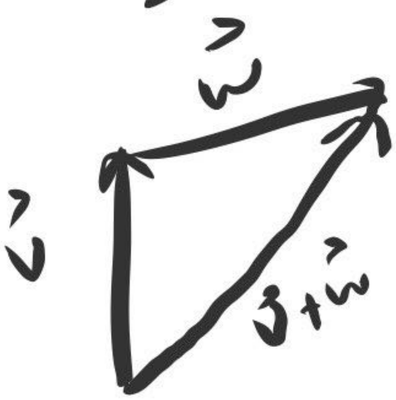
Def. A <u>norm</u> $N : \mathbb{R}^n \to [0, \infty)$

is a function with...

1) $N(\vec{v}) = 0 \Rightarrow \vec{v} = \vec{0}$

2) $N(\lambda \vec{v}) = |\lambda| \, |\vec{v}|$

3) $N(\vec{v} + \vec{w}) \leq N(\vec{v}) + N(\vec{w})$



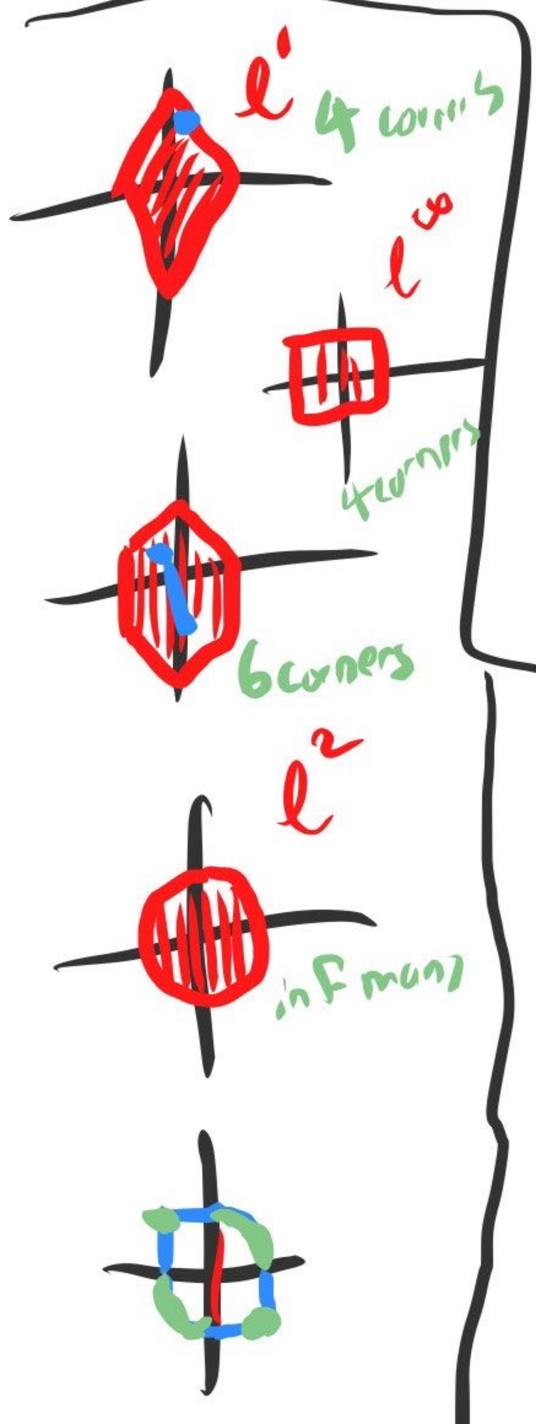
often written $|\vec{v}|$.

This abstract def'n obscures the geometry.

Def. A symmetric convex body in $\mathbb{R}^n$ is a closed set $C \subset \mathbb{R}^n$ s.t. ...

(a) If $\vec{v}, \vec{w} \in C$ then $\lambda \vec{v} + (1-\lambda)\vec{w} \in C$ for $0 \le \lambda \le 1$

(b) If $\vec{v} \in C$ then $-\vec{v} \in C$.



Fact. Giving a <u>norm</u> is the same data as giving a symmetric convex body (w/ nonempty interior).

Upshot: the study of norms is intimately related to "convex geometry".

Def. Two norms $(\mathbb{R}^n, N_1)$ unit ball C_1

$(\mathbb{R}^n, N_2)$ unit ball C_2

are (linearly) <u>isometric</u>

if there is a linear isomorphism

$T : \mathbb{R}^n \to \mathbb{R}^n$

such that $N_2(T(\vec{v})) = N_1(\vec{v})$.

In terms of convex bodies

this means $T(C_1) = C_2$.

So, eg... the norms given by any two ellipsoids are isometric

(in fact, isometric to $\mathbb{R}^n_2$ — standard Euclidean space.)

Not all norms are isometric...

Ex: Invent a good notion of "corners" (extreme point) which only uses scaling & addition in its def'n.

Our cast...

For $1 \le p < \infty$ p real

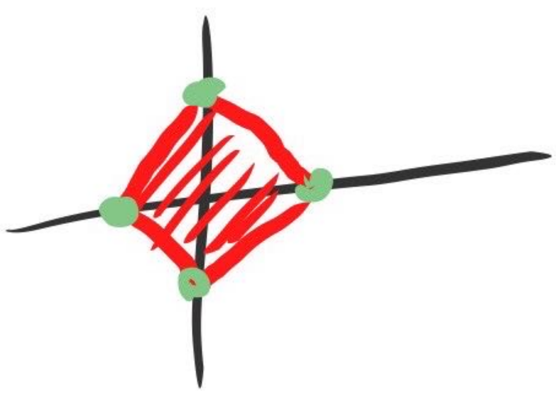
write $|\vec{v}|_p$ for the norm

$$|\vec{v}|_p = \left(|\vec{v}_1|^p + \cdots + |\vec{v}_n|^p\right)^{1/p}$$

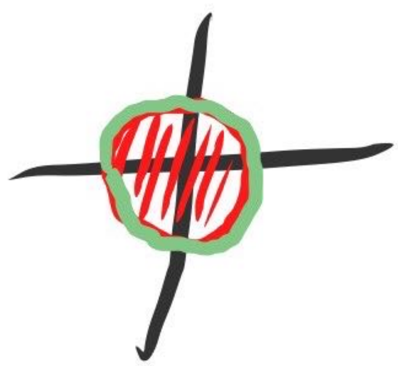
these "limit to"

$$|\vec{v}|_\infty = \max\left(|\vec{v}_1|, \dots, |\vec{v}_n|\right).$$

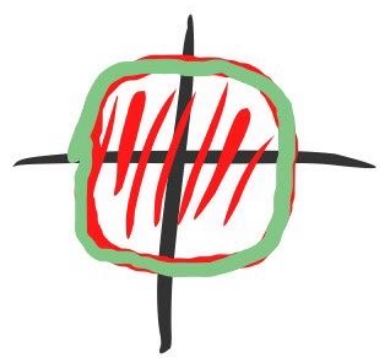
C_1



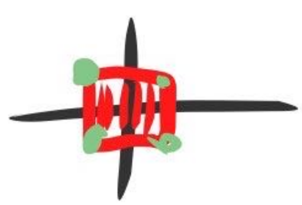
C_2



C_r



C_∞



Today, I'll be telling you why...

ℓ^p is not ℓ^q $\left(\begin{matrix} p \neq q \\ \{p,q\} \neq \{1,\infty\} \end{matrix}\right)$

Thm. ℓ^p is not is ℓ^∞

$(p \neq 1, \infty)$.

Pf: Any linear isometry $T: \mathbb{R}^n_p \to \mathbb{R}^n_\infty$

sends extreme points of C_p ∞ many

$$T^{-1}\uparrow \downarrow T$$

extreme points of C_∞ 4

Contradiction!

But this idea is worthless for $q < \infty$.

But focusing on extreme points quantitatively is.

Spaces of spaces

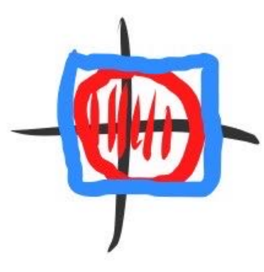
Def. If $T: \mathbb{R}^n_p \to \mathbb{R}^n_q$ is a linear isomorphism, write

$$\|T\|_{p\to q} = \max_{\|\vec{v}\|_p \le 1} \|T\vec{v}\|_q .$$

$\|T\|_{p\to q} \le 1$ iff

$$T(C_p) \subseteq C_q$$

$I: \mathbb{R}^2_2 \to \mathbb{R}^2_\infty$



has $\|I\|_{2\to\infty} = 1$

$\|I\|_{\infty\to 2} = \sqrt{2}$

If T is an isometry

$$T(C_p) = C_q$$

$$T^{-1}(C_q) = C_p$$

then $\|T\|_{p\to q} = \|T^{-1}\|_{q\to p} = 1$

In general $\|T\|_{p\to q} \|T^{-1}\|_{q\to p} \geq 1$

w/ equality iff T is an isometry.

Def. The distance between ℓ^p and ℓ^q

is $\displaystyle\inf_{\substack{\text{linear} \\ \text{iso } T: \mathbb{R}^n \to \mathbb{R}^n}} \log\left(\|T\|_{p\to q} \|T^{-1}\|_{q\to p}\right).$

$d(\ell_p, \ell_q) = 0$ iff ℓ^p, ℓ^q are isometric.

Making the ℓ^∞ vs ℓ^2 argument quantitative.

Thm. $\partial(\mathbb{R}^n_2, \mathbb{R}^n_\infty) = \frac{\log n}{2}$.

Pf: For the identity map

$$I : \mathbb{R}^n_2 \to \mathbb{R}^n_\infty$$

we have $\|I\|_{2\to\infty} = 1$

$$\|I\|_{\infty\to 2} = \sqrt{n}$$

worst at extreme pts.

$$\left(\log\left(\|I\|_{2\to\infty} \|I\|_{\infty\to 2}\right) = \frac{\log n}{2}\right)$$

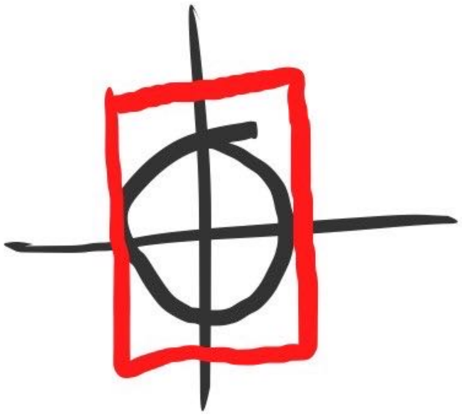
$$\partial(\mathbb{R}^n_2, \mathbb{R}^n_\infty) \le \frac{\log n}{2}$$

Going the other way

also focuses on extreme pts.

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is *any* linear isomorphism,

rescale so $\|T^{-1}\|_{2\to\infty} = 1$



WTS that

$$\|T\|_{\infty \to 2} \geq \sqrt{n}$$

$$\|T^{-1}x\|_\infty \leq \|x\|_2$$

$$\|x\|_\infty \leq \|Tx\|_2$$

I'd like to say... look at $T\begin{pmatrix}\pm 1\\ \vdots \\ \pm 1\end{pmatrix}$.

The same argument calculates exactly

$$d(\mathbb{R}^n_2, \mathbb{R}^n_p) = \log n \left|\frac{1}{2} - \frac{1}{p}\right|$$

Triangle ineq finishes the job (for $p, q \geq 2$)

$$d(\mathbb{R}^n_p, \mathbb{R}^n_q) = \log n \left|\frac{1}{p} - \frac{1}{q}\right|$$

Cor: ℓ^p is not ℓ^2.

Ref: "Handbook of geometry of Banach spaces" Vol 1 Ch 1 §8